Global Supersymmetry on Curved Spaces in Various Dimensions

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Abstract

We provide a classification of d dimensional curved spaces where Euclidean field theories with rigid supersymmetry can be defined. The analysis is carried out from a group theory as well as from a supergravity point of view. In particular, by using appropriate gauged supergravities in various dimensions we show that supersymmetry can be defined in conformally flat spaces, such as non-compact hyperboloids $\mathbb{H}^{n+1}$ and compact spheres $\mathbb{S}^n$ or –by turning on appropriate Wilson lines corresponding to R-symmetry vector fields– on $\mathbb{S}^1 \times \mathbb{S}^n$, with $n < 6$. By group theory arguments we show that Euclidean field theories with rigid supersymmetry cannot be consistently defined on round spheres $\mathbb{S}^d$ if $d > 5$ (despite the existence of Killing spinors). We also show that distorted spheres and certain orbifolds are also allowed by the group theory classification.

1 Introduction

Recently, the study of supersymmetric field theories in Euclidean curved spaces has received considerable attention. In particular, the use of localization techniques has yielded a number of important results, including the exact computation of the partition function, expectation values of Wilson loops and 't Hooft loops in $\mathcal{N} = 2$ theories on $\mathbb{S}^4$ [1–4]. These calculations have been extended to the computation of the partition function of supersymmetric gauge theories on other spaces such as $\mathbb{S}^3$, $\mathbb{S}^1 \times \mathbb{S}^3$, $\mathbb{S}^5$ and some deformed spheres (see e.g. [5–13]). Supersymmetric Yang-Mills (SYM) theories can be defined on these spaces.

A question of interest concerns the classification of all possible Euclidean curved spaces in various dimensions where one can have theories with rigid supersymmetry. In four dimensions, one possible approach [14] is to start with some supergravity theory coupled to matter multiplets in the off-shell formalism. The idea is then to give backgrounds values to the gravity multiplet and to the auxiliary fields that preserve some supersymmetry and then take the limit where the Planck mass goes to infinity. This limit should be taken in a way that the gravitational dynamics decouples and one is left with a theory with rigid supersymmetry on a frozen curved space. This approach has been further developed in many interesting works (see e.g. [15–19], and [20, 21] for earlier studies of rigid superspace geometry). A different interesting approach is in terms of a holomorphic embedding of the space at the boundary of an asymptotically AdS space [22, 23].

The approach based on an off-shell formulation of supergravity is limited to the very few examples where an off-shell formulation is known, for example, $N = 2$ four dimensional supergravity or minimal five-dimensional supergravity. In this paper we shall show that also the on-shell formalism of supergravity can be used to classify the spaces for theories with global supersymmetry. The basic idea is as follows. One starts with any (Euclidean) supergravity action in d dimensions, give background values to the gravity multiplet and possibly to other multiplets. The resulting action will be supersymmetric if supersymmetry transformations do not change these values. This requires that supersymmetry transformations on all fermions vanish. In particular, the vanishing of the gravitino shift gives an equation for the Killing spinor on a specific gravitational background. If a solution for the Killing spinor exists, then there is some remaining supersymmetry (barring certain subtleties that appear in dimensions $d > 5$ –discussed in sections 5 and 6). This is in principle enough for the problem of classification of supersymmetric curved spaces studied in this paper.

Having identified a given supersymmetric space, the next problem concerns the determination of

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the desired field theory Lagrangian with global supersymmetry. Although finding specific field theory Lagrangians goes beyond the scope of this work, in section 3.3 we briefly comment on the prescription that one would have to follow within the on-shell approach. The decoupling of gravity must be done in the usual way by taking the limit where the Planck mass goes to infinity. Obtaining non-trivial curved spaces require that, at the same time, background values of matter fields are sent to infinity in an appropriate way. As long as the limit is regular, one is left with a field theory on a curved space which, by construction, has global supersymmetry.

In our quest for the classification of supersymmetric curved spaces, we will follow two different ways. The first one is based entirely on group theory. In fact, group theory already gives some model independent results in Poincaré supersymmetry. We may recall for example that group theory arguments restrict the maximal number of spacetime dimensions to eleven for a supersymmetric theory with one time direction and a single graviton [24]. Similarly, all manifest supersymmetries in different dimensions, including those with conformal or de Sitter space-time symmetry, have been determined and, in particular, all possible simple supersymmetries have been classified [24]. This gives us the possibility of identifying group theoretically all possible spaces admitting as isometry groups the bosonic part of the allowed supersymmetry groups. The second way of classifying supersymmetric spaces is based on on-shell supergravity as explained above.

In the next section, we begin by providing a classification of supersymmetric spaces in various dimensions based on group theory arguments. In section 3 and section 4 we discuss the 4D supergravity and the $N = 2$ 5d supergravity. In section 5 the 6d $F(4)$ supergravity [25] is considered and in section 6 we generally comment on supersymmetry on $d > 5$ spaces.

2 Group Theoretic Approach

Supersymmetry generators form a superalgebra [26–28]. The latter has a graded $\mathbb{Z}_2$ structure which splits its generators into even and odd parts. The even generators form a classical algebra whereas the odd part transforms under some representation of the even part. In supersymmetry algebra the odd part is in the spinorial representation of its even (bosonic) part. In fact, all possible simple supersymmetry algebras have been classified long ago by Nahm [24]. By splitting the superalgebra $\mathcal{G}$ in the even G_0 and odd G_1 parts as $\mathcal{G} = G_0 \oplus G_1$ with generators of G_1 transforming in the R representation of G_0, the possible superalgebras (in the Euclidean regime) are the following:

I. $\mathcal{G} = F(4), \quad G = SO(6,1) \oplus SU(2), \quad R = (8,2)$. $\hspace{3cm}$ (2.1)

This case can describe a supersymmetric theory on the hyperbolic space $\mathbb{H}^6$ or a superconformal theory on $\mathbb{S}^5$.

II. $\mathcal{G} = su(4|N), \quad G = SO(6) \oplus U(N), \quad R = (4,N) + (\bar{4},\overline{N})$, $\quad N \neq 4$,

$\qquad \mathcal{G} = su(4|4), \quad G = SO(6) \oplus SU(4), \quad R = (4,4) + (\bar{4},\bar{4})$, $\hspace{2cm}$ (2.2)

for a supersymmetric theory on the round $\mathbb{S}^5$.

III. $\mathcal{G} = su^*(4|2N)^1$, $\quad G = SO(5,1) \oplus U(2N)$, $\quad R = (4,2N) + (\bar{4},\overline{2N})$, $\hspace{1.5cm}$ (2.3)

for a superconformal theory on round $\mathbb{S}^4$ or a supersymmetric theory on the hyperbolic space $\mathbb{H}^5$.

IV. $\mathcal{G} = osp(2|4), \quad G = SO(5) \oplus U(1), \quad R = 4 + \bar{4}$, $\hspace{3cm}$ (2.4)

for a supersymmetric theory on the round $\mathbb{S}^4$.

V. $\mathcal{G} = osp(2|2,2), \quad G = SO(4,1) \oplus U(1), \quad R = 4 + \bar{4}$, $\hspace{2.5cm}$ (2.5)

for a superconformal theory on round $\mathbb{S}^3$ or a supersymmetric theory on the hyperbolic space $\mathbb{H}^4$.

VI. $\mathcal{G} = su(2|N) \oplus su(2|N), \quad G = SO(4) \oplus U(N)^2, \quad R = (2,1,N,1) + (1,2,1,N), \quad N \neq 2$, $\hspace{0.5cm}$ (2.6)

or $\quad \mathcal{G} = su(2|2) \oplus su(2|2), \quad G = SO(4) \oplus SU(2)^2, \quad R = (2,2,1,1) + (1,1,2,2)$, $\hspace{1cm}$ (2.7)

for a supersymmetric theory on round $\mathbb{S}^3$.

VII. $\mathcal{G} = osp(3,1), \quad G = SO(3,1), \quad R = (2,1) + (1,2)$, $\hspace{3cm}$ (2.8)

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[1]This case is missing from [24] but appears in [29] as $u_E(4,N)$.

for a superconformal theory on round $\mathbb{S}^2$ or a supersymmetric theory on the hyperbolic space $\mathbb{H}^3$.

VIII. $\quad \mathcal{G} = su(2|N), \quad G = SO(3) \oplus U(N), \quad R = (2,N) + (2,\bar{N})$,

$\qquad \mathcal{G} = su(2|2), \quad G = SO(3) \oplus SU(2), \quad R = (2,2) + (2,2)$, $\qquad\qquad\qquad$ (2.9)

for a supersymmetric theory on the round $\mathbb{S}^2$.

IX. $\quad \mathcal{G} = su(1,1|N), \quad G = SO(2,1) \oplus U(N), \quad R = (2,N) + (2,\bar{N})$,

$\qquad \mathcal{G} = su(1,1|2), \quad G = SO(2,1) \oplus SU(2), \quad R = (2,2) + (2,2)$, $\qquad\qquad$ (2.10)

X. $\quad \mathcal{G} = osp(N|2), \quad G = SO(2,1) \oplus SO(N), \quad R = (2,N)$, $\qquad\qquad\qquad$ (2.11)

and

XI. $\quad \mathcal{G} = osp(4|2,a), \quad G = SO(2,1) \oplus O(4), \quad R = (2,4)_a$, $\qquad\qquad\qquad$ (2.12)

for supersymmetric theories on the hyperboloid $\mathbb{H}^2$, or superconformal theories on $\mathbb{S}^1$.

XII. $\quad \mathcal{G} = F(4), \quad G = SO(2,1) \oplus SO(7), \quad R = (2,8)$, $\qquad\qquad\qquad$ (2.13)

XIII. $\quad \mathcal{G} = G(3), \quad G = SO(2,1) \oplus G_2, \quad R = (2,7)$, $\qquad\qquad\qquad$ (2.14)

for supersymmetric theories on the hyperboloid $\mathbb{H}^2$, or superconformal theories on $\mathbb{S}^1$.

XIV. $\quad \mathcal{G} = osp(2|2N), \quad G = SO(2) \oplus Sp(2N), \quad R = 2N \oplus 2N$, $\qquad\qquad$ (2.15)

for a supersymmetric theory on $\mathbb{S}^1$.

We have collected the above in Table 1 and represented the cases of simply connected, maximally symmetric spaces admitting supersymmetric theories. It should be noted that round spheres $\mathbb{S}^d$ with $d > 5$, and hyperboloids $\mathbb{H}^d$ with $d > 6$, are not allowed by the Nahm classification. We will further comment on this in section 5 and section 6.

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$\mathcal{G}$	G_0	R	SUSY	SC
$osp(6,1\|2)$	$SO(6,1)\oplus SU(2)$	(8,2)	$\mathbb{H}^6$	$\mathbb{S}^5$
$su(4\|N)$	$SO(6)\oplus U(N)$	$(4,N)+(\bar{4},\bar{N}), \quad N\neq 4$	$\mathbb{S}^5$	
$su(4\|4)$	$SO(6)\oplus SU(4)$	$(4,4)+(\bar{4},\bar{4})$	$\mathbb{S}^5$	
$su^*(4\|2N)$	$SO(5,1)\oplus U(2N)$	$(4,2N)+(\bar{4},\bar{2N})$	$\mathbb{H}^5$	$\mathbb{S}^4$
$osp(2\|4)$	$SO(5)\oplus U(1)$	$4+\bar{4}$	$\mathbb{S}^4$	
$osp(2\|2,2)$	$SO(4,1)\oplus U(1)$	$4+\bar{4}$	$\mathbb{H}^4$	$\mathbb{S}^3$
$su(2\|N)\oplus su(2\|N)$	$SO(4)\oplus U(N)^2$	$(2,N,1,1)+(1,1,2,N)$	$\mathbb{S}^3$	
$su(2\|2)\oplus su(2\|2)$	$SO(4)\oplus SU(2)^2$	$(2,2,1,1)+(1,1,2,2)$	$\mathbb{S}^3$	
$su(2\|N)$	$SO(3)\oplus U(N)$	$(2,N)+(2,\bar{N})$	$\mathbb{S}^2$	
$osp(3\|2)$	$SO(3)\oplus SU(2)$	$(2,2)+(2,\bar{2})$	$\mathbb{S}^2$	
$osp(3,1)$	$SO(3,1)$	$(2,1)+(1,2)$	$\mathbb{H}^3$	$\mathbb{S}^2$
$su(1,1\|N)$	$SO(2,1)\oplus U(N)$	$(2,N)+(2,\bar{N})$	$\mathbb{H}^2$	$\mathbb{S}^1$
$su(1,1\|2)$	$SO(2,1)\oplus SU(2)$	$(2,2)+(2,\bar{2})$	$\mathbb{H}^2$	$\mathbb{S}^1$
$osp(N\|2)$	$SO(2,1)\oplus SO(N)$	$(2,N)$	$\mathbb{H}^2$	$\mathbb{S}^1$
$osp(4\|2,a)$	$SO(2,1)\oplus O(4)$	$(2,4)_a$	$\mathbb{H}^2$	$\mathbb{S}^1$
$F(4)$	$SO(2,1)\oplus SO(7)$	$(2,8)$	$\mathbb{H}^2$	$\mathbb{S}^1$
$G(3)$	$SO(2,1)\oplus G_2$	$(2,7)$	$\mathbb{H}^2$	$\mathbb{S}^1$
$osp(2\|2N)$	$SO(2)\oplus Sp(2N)$	$2N\oplus 2N$	$\mathbb{S}^1$	

Table 1: The superalgebras for maximally symmetric spaces.

Dimension	Background	Superalgebra
2	$\mathbb{S}^1\times\mathbb{S}^1$	$\mathcal{G}_1\oplus\mathcal{G}_1$
		$osp(2\|2N)\oplus\mathcal{G}_2$
3	$\mathbb{S}^1\times\mathbb{S}^2$	$osp(3,1)\oplus\mathcal{G}_1$
		$su(2\|N)\oplus\mathcal{G}_2$
4	$\mathbb{S}^1\times\mathbb{S}^3$	$osp(2\|2,2)\oplus\mathcal{G}_1$
		$su(2\|N)\oplus su(2\|N)\oplus\mathcal{G}_2$
5	$\mathbb{S}^1\times\mathbb{S}^4$	$su^*(4\|2N)\oplus\mathcal{G}_1$
		$osp(2\|4)\oplus\mathcal{G}_2$
6	$\mathbb{S}^1\times\mathbb{S}^5$	$osp(6,1\|2)\oplus\mathcal{G}_1$
		$su(4\|N)\oplus\mathcal{G}_2$

Table 2: Some interesting product spaces and their corresponding superalgebra are given by the (non-exhaustive) list above. $\mathcal{G}_1$ is one of $\{su(2\|N), osp(N\|2), osp(4\|2,a), F(4), G(3)\}$ and $\mathcal{G}_2$ is one of $(su(2\|N), osp(N\|2), osp(4\|2,a), F(4), G(3), osp(2\|2N))$.

Product spaces are also possible. They correspond to non-simple superalgebras and some interesting cases are presented in Table 2. Note, however, as we will see in the next sections, only products with $\mathbb{S}^1$

factors are possible as these products preserve conformal flatness.

2.1   Ellipsoids

In a similar manner one can also have superalgebras on ellipsoids in several dimensions. They are generally described by the equation

$$\sum_{\mu=1}^{d} \frac{x_\mu^2}{\ell_\mu^2} = 1 \ . \tag{2.16}$$

The superalgebra classification is determined in terms of the bosonic isometries. Examples are given below:

- Ellipsoids preserving only $U(1)$ isometries allow for superalgebras of the form

$$\mathcal{G} = osp(2|2N) \oplus ... \oplus osp(2|2N) \ ,$$

  i.e. direct sums of type XIV superalgebras.

- Ellipsoids preserving an $SO(3)$ isometry and a $U(1)$ isometry allow for superalgebras of the form

$$\mathcal{G} = osp(3|N) \oplus osp(2|2N) \ ,$$

  i.e. it is a direct sum of a type VIII and type XIV superalgebras.

- Ellipsoids preserving an $SO(4)$ isometry and a $U(1)$ isometry allow for superalgebras of the form

$$\mathcal{G} = sl(2|N) \oplus sl(2|N) \oplus osp(2|2N) \ .$$

- Ellipsoids preserving an $SO(5)$ isometry and a $U(1)$ isometry allow for superalgebras of the form

$$\mathcal{G} = osp(2|4) \oplus osp(2|2N) \ .$$

2.2   Orbifolds

There is another class of manifold which admits supersymmetry, and is connected to the cases I–XIV described above. Their construction is as follows. Consider a manifold N with isometry group G and let $\Gamma \subset G$ a discrete subgroup of the latter freely acting on N. Then the space $M = N/\Gamma$ is non-singular and corresponds to global identifications of N. Therefore, the isometries of M will be different from

those of N leading to different supersymmetry algebra supported by M. To be precise, let us consider in particular $\mathbb{S}^3$, as the other cases are quite similar.

The group of orientation-preserving isometries of $\mathbb{S}^3$ is $SO(4)$. The quotient of $SO(4)$ by its center $\{\pm I\}$ is isomorphic to $SO(3) \times SO(3)$, therefore a finite subgroup G of $SO(4)$ gives rise to two finite subgroups G_L and G_R of $SO(3)$. These considerations specify the possible finite subgroups of $\mathbb{S}^3$ to be

$$\Gamma = \mathbb{Z}_n\,, \quad D_{4n}^*\,, \quad T_{24}^*\,, \quad O_{48}^*,,\quad I_{120}^*\,, \tag{2.17}$$

i.e., the finite cyclic groups $\mathbb{Z}_n$, the binary dihedral groups D_{4n} of order 4n, and the binary tetrahedral, octahedral, and icosahedral groups T_{24}^*, O_{48}^*, and I_{120}^*, of orders 24, 48, and 120, respectively. If G acts freely on $\mathbb{S}^3$, then, say, G_L must be cyclic, and G_R can then be described as being of cyclic, dihedral, tetrahedral, octahedral, or icosahedral type, according to the type of G_R. The groups of cyclic type are cyclic, and the corresponding 3-manifolds are the lens spaces $L(m, n)$ defined by the identification

$$(z_1, z_2) \equiv (e^{2\pi i/m} z_1, e^{2\pi i n/m} z_2) \tag{2.18}$$

of the coordinates of (z_1, z_2) of $\mathbb{C}^2$ where $\mathbb{S}^3$ is embedded. The isometry groups of $\mathbb{S}^3/\Gamma$ have been calculated in [30]. Here we just mention that for lens spaces we have for example that

$$\texttt{isom}\big(\mathbb{S}^3/T_{24}^*\big) = SO(3) \times \mathbb{Z}_2\,, \quad \texttt{isom}\big(\mathbb{S}^3/T_{24}^* \times \mathbb{Z}_2\big) = SO(2) \times \mathbb{Z}_2 \tag{2.19}$$

where $\texttt{isom}(M)$ is the isometry group of M. Similarly, for the lens spaces $L(m, 1) = \mathbb{S}^3/Z_m$ and $L(m, n)$ we have

$$\texttt{isom}\big(\mathbb{S}^3/Z_m\big) = SO(3) \times U(1)\,, \qquad m > 2$$
$$\texttt{isom}\big(L(m, n)\big) = U(1) \times U(1) \qquad m,\, (n^2 - 1)/m\,, \text{ even} \tag{2.20}$$

These isometry groups will then be the even part of the supersymmetry algebra on $\mathbb{S}^3/\Gamma$. For example, let us take the case of $\mathbb{S}^3/Z_m$. Starting with the round $\mathbb{S}^3$, we may consider the simplest case of an $osp(4|2n)$ superalgebra with bosonic subgroup $G_0 = SU(2) \times SU(2) \times Sp(2n)$. By embedding the $\mathbb{Z}_m$ in the $U(1)$ subgroup of the second $SU(2)$, the isometry group is broken down to $SU(2) \times U(1)$. Similarly, the fermionic generators are in the $(2, 1, 2n) + (1, 2, 2n)$ representation of G_0 and only the $(2, 1, 2n)$ part survives the modding by $\mathbb{Z}_m$. Therefore, the supersymmetry algebra for the $\mathbb{S}^3$ has been reduced to $osp(3|2n)$. Similar considerations apply to the other cases and in higher dimensional spheres as well. The partition function for super Yang-Mills theory on $\mathbb{R} \times \mathbb{S}^3/\mathbb{Z}_m$ has been recently computed in [13].

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3 Supersymmetric 4d spaces from matter superfields coupled to supergravity

3.1 $N = 1$ supergravity

In [14,17,18], supersymmetric spaces are obtained by giving background values to auxiliary fields. In this approach, the auxiliary fields are not required to satisfy the equations of motion. The curved geometries are then supported by the background auxiliary fields. It is interesting to see how the different possible curved spaces are realized in the on-shell formalism, where auxiliary fields have already been eliminated by their equations of motion. In this approach one has to give background values to some dynamic fields.

In this subsection we first discuss in detail the case of old minimal $N = 1$ supergravity with Euclidean signature coupled to chiral superfields, and then consider the addition of vector multiplets. Supersymmetry requires that the supersymmetry transformations on fermions vanish on a given bosonic background. In the Euclidean theory the left and right handed components of a fermion $P_L\psi$ and $P_R\psi$ will be independent. We will denote them by ψ and $\bar\psi$. The supersymmetry transformations for the left and right handed gravitino are (see (18.22) in [31])

$$\delta\psi_\mu = \nabla_\mu\xi + iA_\mu\xi + \frac{1}{2}B\Gamma_\mu\bar\xi \ , \tag{3.1}$$

$$\delta\bar\psi_\mu = \nabla_\mu\bar\xi - iA_\mu\bar\xi + \frac{1}{2}\bar{B}\Gamma_\mu\xi \ , \tag{3.2}$$

where

$$A_\mu \equiv -\frac{i\kappa^2}{4}\big(K_\alpha\partial_\mu\phi^\alpha - K_{\bar\alpha}\partial_\mu\bar\phi^{\bar\alpha}\big) \ , \qquad B \equiv \kappa^2\ e^{\kappa^2 K/2}\ W \ , \tag{3.3}$$

and

$$\nabla_\mu\xi = \big(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\big)\xi \ , \qquad \Gamma_{ab} = \frac{1}{2}\big(\Gamma_a\Gamma_b - \Gamma_b\Gamma_a\big) \ .$$

As usual, $W = W(\phi^\alpha)$, $\alpha = 1,...,n_c$, denotes the superpotential and $K = K(\phi^\alpha, \bar\phi^\alpha)$ is the Kähler potential. μ, ν and a, b respectively denote curved space and tangent space indices. We shall follow the conventions of [31] for spinors and Dirac Γ matrices. We also need to set to zero the supersymmetry variations of the fermions of the chiral multiplets, $\delta\chi^\alpha = \delta\bar\chi^\alpha = 0$. They will be discussed below.

Consider the equations $\delta\psi_\mu = \delta\bar\psi_\mu = 0$. The integrability condition for the equation $\delta\psi_\mu = 0$ gives

$$\begin{aligned} 0 &= \big[\nabla_\nu, \nabla_\mu\big]\xi + iF_{\nu\mu}\ \xi + iA_\mu\nabla_\nu\xi - iA_\nu\nabla_\mu\xi \\ &+ \frac{1}{2}\big(\nabla_\nu B\Gamma_\mu\bar\xi - \nabla_\mu B\Gamma_\nu\bar\xi + B\Gamma_\mu\nabla_\nu\bar\xi - B\Gamma_\nu\nabla_\mu\bar\xi\big) \ . \end{aligned} \tag{3.4}$$

Using the equations for ξ, $\bar{\xi}$, this becomes

$$
\begin{aligned}
0 \;=\;& \frac{1}{4} R_{\nu\mu ab} \Gamma^{ab} \xi + i F_{\nu\mu} \, \xi - \frac{1}{2} B \bar{B} \Gamma_{\mu\nu} \\
& - \frac{1}{2} \left[\Gamma_\nu \left(\nabla_\mu B + 2 i B A_\mu \right) - \Gamma_\mu \left(\nabla_\nu B + 2 i B A_\nu \right) \right] \; .
\end{aligned}
\tag{3.5}
$$

Therefore, we get the conditions

$$
F_{\mu\nu} = 0 \; , \qquad \nabla_\mu B = -2 i B A_\mu \; , \qquad \nabla_\mu \bar{B} = 2 i \bar{B} A_\mu \; ,
\tag{3.6}
$$

$$
R_{\nu\mu ab} \Gamma^{ab} \xi = 2 B \bar{B} \Gamma_{\mu\nu} \xi \; .
\tag{3.7}
$$

This last equation implies that

$$
R_{\mu\nu\rho\sigma} = -B\bar{B} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right) \; ,
\tag{3.8}
$$

i.e. the space is locally isometric to a maximally symmetric space. Therefore the Weyl tensor $W_{\mu\nu\rho\sigma}$ vanishes and the space is conformally flat. We have

$$
W_{\mu\nu\rho\sigma} = 0 \; , \qquad R_{\mu\nu} = -3 B \bar{B} \; g_{mn} \; .
\tag{3.9}
$$

Let us now examine the conditions (3.6) in detail. One solution is

$$
B = 0 \; , \qquad A_\mu = \text{arbitrary} \; .
\tag{3.10}
$$

This gives flat space-time.

If $B \neq 0$, then (3.6) is solved by an A_μ of the form, $A_\mu = \nabla_\mu \Lambda$. However, from the definition (3.3) of A_μ, the integrability condition then leads to $K_{\alpha\bar{\beta}} d\phi^\alpha \wedge d\phi^{\bar{\beta}} = 0$, which implies $\phi^\alpha = \phi_0^\alpha = \text{constant}$, hence $A_\mu = 0$. Therefore the unique solution is

$$
A_\mu = 0 \; , \qquad B = B_0 = \text{const} \; .
\tag{3.11}
$$

We still need to check the equation for the fermions of the chiral multiplets. For constant scalars, this gives

$$
\begin{aligned}
\delta\chi^\alpha \;=\;& -\frac{1}{\sqrt{2}} e^{\frac{\kappa^2 K}{2}} (K^{\alpha\bar{\beta}} \bar{\nabla}_{\bar{\beta}} \bar{W}) \, \xi = 0 \; , \\
\delta\bar{\chi}^{\bar{\alpha}} \;=\;& -\frac{1}{\sqrt{2}} e^{\frac{\kappa^2 K}{2}} (K^{\bar{\alpha}\beta} \nabla_\beta W) \, \bar{\xi} = 0 \; .
\end{aligned}
\tag{3.12}
$$

Thus we get the following condition for the constant values ϕ_0, $\bar{\phi}_0$ of scalar fields:

$$\nabla_\alpha W\Big|_{\phi_0} = \bar{\nabla}_{\bar{\alpha}} \bar{W}\Big|_{\phi_0} = 0 \ . \tag{3.13}$$

Let us now consider the addition of vector multiplets. We now need to impose, in addition, that the supersymmetry transformation of the gauginos vanishes. Assuming constant values for the vector bosons, this gives the extra condition

$$\delta\lambda^A = \frac{i}{2}\gamma_5 (\mathrm{Re}f)^{-1\ AB}\mathcal{P}_B\xi = 0 \ , \tag{3.14}$$

where $f_{AB}(\phi)$ are the holomorphic functions determining the gauge multiplet kinetic terms and $\mathcal{P}_A(\phi,\bar{\phi})$ are the Killing potentials. This implies

$$\mathcal{P}_A(\phi_0,\bar{\phi}_0) = 0 \ . \tag{3.15}$$

This is an extra condition on the constant background values ϕ_0^α, $\bar{\phi}_0^\alpha$. The gravitino transformation is not changed. Therefore, turning background values for vector bosons (Wilson lines) does not generate new supersymmetric spaces.

Thus we find Einstein-Weyl spaces of negative curvature. This implies that the space is locally isometric to $\mathbb{H}^4$. Supersymmetric theories on a positive curved space $\mathbb{S}^4$ can be obtained by relaxing the condition that $\bar{B}$ is the complex conjugate of B. As discussed in an analogous context in [14], this leads to a Lagrangian which is not real. The resulting Euclidean theory is not reflection positive and does not correspond to any unitary theory with Lorentzian signature. This is not surprising, since general supersymmetric theories cannot be put on dS_4. An exception occurs when the theory is superconformal; then dS_4 is admitted, as this space is conformal to Minkowski space. In this case the Lagrangian becomes real.

It should be noted that, for superconformal theories, more general spaces are allowed. Indeed, they can be formulated in any space which is conformal to $\mathbb{H}^4$ (or, equivalently, to $\mathbb{S}^4$). This gives more options, in particular, spaces of the form $X \times \mathbb{S}^1$, where X is locally isometric to a maximally symmetric space. If the theory is not superconformal, the above calculation shows that it cannot be put on any non-Einstein space, in particular, in a curved space $X \times \mathbb{S}^1$, by just turning on background values for scalar fields or vector fields not associated with an R-symmetry. We next show that spaces of the form $X \times \mathbb{S}^1$ can be admitted if a suitable Wilson line background field corresponding to an R-symmetry is turned on in the $\mathbb{S}^1$ direction.

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3.2 $N = 2$ four-dimensional gauged supergravity

Here we shall show that theories with rigid supersymmetry can be formulated on spaces of the form $X \times \mathbb{S}^1$, where X is locally isometric to a maximally symmetric space, by turning on suitable R-symmetry vector field components in the $\mathbb{S}^1$ direction.

The mechanism can be implemented in any dimension $d \leq 6$, starting with a suitable gauged super-gravity. Different dimensions need to be examined case by case, because there are important technical differences, in particular due to the different gauge groups and spinor representations. In this section we begin by considering the four-dimensional case.

Our starting point is $\mathcal{N} = 2$ gauged supergravity coupled to any number of vector and hyper multiplets. For a detailed description of the theory we refer to [31]. The scalar manifold is a direct product of the special Kähler manifold of the scalars in the vector multiplets and the quaternionic Kähler manifold of the scalars in the hypermultiplets. The graviton multiplet contains the vierbein e_μ^a, two gravitinos ψ_μ^i and the graviphoton A_μ^0. By turning on constant values for the complex scalar fields z^α, $\bar{z}^{\bar\alpha}$ of the vector multiplet ($\alpha = 1, ..., n_V$), constant values for the real scalars q^u of the hypermultiplets ($u = 1, ..., 4n_H$), and constant values for vector field components A_μ^I ($I = 0, 1, ..., n_V$), the supersymmetry transformation law for the left and right handed gravitinos take the form (see [31], eq. (21.42)),

$$\delta\psi_\mu^i = \nabla_\mu \xi^i - \frac{i}{2}\mathcal{A}_\mu \xi^i + \mathcal{V}_\mu{}^i{}_j \xi^j + \frac{1}{2}\kappa^2 \Gamma_\mu S^{ij}\xi_j = 0 \ , \tag{3.16}$$

where

$$\mathcal{A}_\mu = -\kappa^2 A_\mu^I P_I^0 \ , \qquad \mathcal{V}_\mu^{ij} = -\frac{\kappa^2}{2} A_\mu^I P_I^{ij} \ , \qquad S^{ij} = P_I^{ij}\bar{X}^I \ , \tag{3.17}$$

and $P_I^{ij}(q)$ denote, as usual, moment maps on the quaternionic Kähler metric g_{XY} of the hypermultiplet scalar manifold, and $P_I^0(z, \bar{z})$ is the real moment map of the special Kähler manifold. S^{ij} is a symmetric matrix which depends on the constant background values for the scalars q, $\bar{z}$. We recall that spinors are $SU(2)$ doublets and $SU(2)$ indices are lowered and raised by ϵ_{ij}.

To simplify the discussion in what follows we assume that $\mathcal{V}_\mu{}^i{}_j = 0$, since it is not needed to generate the relevant solutions.

The supersymmetry parameter ξ_i is a symplectic Majorana spinor satisfying

$$\xi_i = \xi^j \epsilon_{ji} \ , \qquad \xi_i = (\xi^i)^C \ , \tag{3.18}$$

where λ^C denotes charge conjugation. Like in the $N = 1$ case, in Euclidean space we must relax this condition and treat ξ^i and $\bar{\xi}^i \equiv (\xi^i)^C$ as independent.[2]

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[2]Supersymmetry in Euclidean space is an old subject on its own [32]. In Euclidean space the charge conjugation

Thus we have two independent equations

$$\delta\psi^i_\mu = \nabla_\mu\xi^i - \frac{i}{2}\mathcal{A}_\mu\xi^i + \frac{1}{2}\kappa^2\Gamma_\mu S^{ij}\bar{\xi}^j = 0 \ ,$$

$$\delta\bar{\psi}^i_\mu = \nabla_\mu\bar{\xi}^i + \frac{i}{2}\bar{\mathcal{A}}_\mu\bar{\xi}^i + \frac{1}{2}\kappa^2\Gamma_\mu\bar{S}^{ij}\xi^j = 0 \ . \tag{3.19}$$

Likewise, S^{ij} and $\bar{S}^{ij}$ and $\mathcal{A}_\mu$ and $\bar{\mathcal{A}}_\mu$ will be treated as independent. This doubling of boson degrees of freedom in Euclidean space is in consistency with the doubling of fermion degrees of freedom. The supersymmetry variations of other fermions will be discussed below.

As in the $N = 1$ case, the integrability condition of (3.19) implies that

$$F_{\mu\nu} = 0 \ . \tag{3.20}$$

which is automatically satisfied for constant $\mathcal{A}_\mu$. We are left with

$$0 = [\nabla_\nu, \nabla_\mu]\xi^i - \frac{\kappa^4}{2}S^{ij}\bar{S}^{jk}\Gamma_{\mu\nu}\xi^k + \frac{i\kappa^2}{2}(\mathcal{A}_\mu\Gamma_\nu - \mathcal{A}_\nu\Gamma_\mu)S^{ij}\bar{\xi}^j \ ,$$

$$0 = [\nabla_\nu, \nabla_\mu]\bar{\xi}^i - \frac{\kappa^4}{2}\bar{S}^{ij}S^{jk}\Gamma_{\mu\nu}\bar{\xi}^k - \frac{i\kappa^2}{2}(\bar{\mathcal{A}}_\mu\Gamma_\nu - \bar{\mathcal{A}}_\nu\Gamma_\mu)\bar{S}^{ij}\xi^j \ . \tag{3.21}$$

We will assume that $[S, \bar{S}] = 0$. Let us now consider the supersymmetry variation of the gauginos and hyperinos. Analogously to the $N = 1$ case, these lead to algebraic constraints on the constant values of the scalar fields. In the notation of [31]

$$\bar{W}_{\bar{\beta}ji}(z, \bar{z}, q) = W_{\beta ji}(z, \bar{z}, q) = 0 \ , \qquad \bar{N}^A_i(z, \bar{z}, q) = N^A_i(z, \bar{z}, q) = 0 \ . \tag{3.22}$$

Solving these equations explicitly requires specifying the model. Note that these points correspond to fixed points of the scalar manifold in supersymmetric flows. The standard relation that connects the scalar potential V to fermion shifts now gives the identity

$$-3\kappa^2 S^{ik}\bar{S}_{jk} = \delta^i_j \ V \tag{3.23}$$

If $\mathcal{A}_\mu = 0$, the gravitino equation can be solved without any restriction on the spinors. It implies

$$W_{\mu\nu\rho\sigma} = 0 \ , \qquad R_{\mu\nu} = \kappa^2 V g_{\mu\nu} \ . \tag{3.24}$$

i.e., the space is Einstein with vanishing Weyl tensor. The choices $V < 0$ and $V > 0$ respectively give spaces locally isometric to $\mathbb{H}^4$ and $\mathbb{S}^4$.

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matrix has imaginary eigenvalues and the Majorana condition cannot be imposed, although there are alternative treatments (see e.g. [33]).

Consider now a reducible space of the form $X_3 \times \mathbb{S}^1$. As this space has non-trivial homotopy group $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ one can turn on Wilson lines. Then one can find the following solution. We turn on constant components $\mathcal{A}_4$, $\bar{\mathcal{A}}_4$ and $\mathcal{A}_{\hat{\mu}} = \bar{\mathcal{A}}_{\hat{\mu}} = 0$, $\hat{\mu} = 1, 2, 3$, and constant scalars. The vanishing of the gaugino and hyperino variations again imply the conditions (3.22). Consider now the gravitino variation. It is convenient to consider a spinor basis ξ, $\bar{\xi}$ where Γ_4 is diagonal. We demand,

$$\partial_4 \xi_i = \partial_4 \bar{\xi}_i = 0 . \tag{3.25}$$

The fourth component of the conformal Killing spinor equation (3.19) then reads

$$i \mathcal{A}_4 \xi^i = \kappa^2 \Gamma_4 S^{ij} \bar{\xi}^j , \qquad i \bar{\mathcal{A}}_4 \bar{\xi}^i = -\kappa^2 \Gamma_4 \bar{S}^{ij} \xi^j . \tag{3.26}$$

Equation (3.26) reduces the number of supersymmetries by a factor of $1/2$. In particular, if the first equation is solved for the spinors with $\Gamma_4 = 1$, the spinors with $\Gamma_4 = -1$ must be set to zero. Combining both equations, we find

$$\mathcal{A}_4 \bar{\mathcal{A}}_4 = -\frac{\kappa^2}{3} V , \tag{3.27}$$

with no further restriction on the spinors. Substituting into the remaining equations, we find

$$\nabla_{\hat{\mu}} \xi^i = \frac{i}{2} \mathcal{A}_4 \Gamma_4 \Gamma_{\hat{\mu}} \xi^i , \qquad \nabla_{\hat{\mu}} \bar{\xi}^i = -\frac{i}{2} \bar{\mathcal{A}}_4 \Gamma_4 \Gamma_{\hat{\mu}} \bar{\xi}^i . \tag{3.28}$$

The integrability conditions of these equations imply that

$$W_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}(X) = 0 , \qquad R_{\hat{\mu}\hat{\nu}} = \frac{2\kappa^2}{3} V \, g_{\hat{\mu}\hat{\nu}} , \tag{3.29}$$

$$\mathcal{A}_4^2 = \bar{\mathcal{A}}_4^2 = -\frac{\kappa^2}{3} V . \tag{3.30}$$

The Einstein-Weyl condition implies that X_3 is locally isometric to a maximally symmetric space. Therefore we get spaces $X_3 \times \mathbb{S}^1$ by turning on a Wilson line. According to the sign of V, we find spaces locally isometric to $\mathbb{H}^3 \times \mathbb{S}^1$ or $\mathbb{S}^3 \times \mathbb{S}^1$.

Finally, note that the backgrounds discussed so far do not allow for supersymmetric theories on ellipsoids. Killing spinors on ellipsoids can be obtained by turning on suitable $SU(2)_R$ gauge fields and tensor fields [10]. This suggests that it might be possible to obtain theories with rigid supersymmetry on ellipsoids from on-shell $N = 4$ gauged supergravity. It would be interesting to construct the explicit solution of the Killing spinor equations in this way.

3.3 Comments on the construction of supersymmetric Lagrangians

The off-shell treatment in supergravity is limited to few examples such as $N = 2$ four dimensional supergravity or minimal five-dimensional supergravity. Therefore, it is important to understand the construction of supersymmetric Lagrangians within the on-shell formalism.

Consider first the $N = 1$ four-dimensional case discussed in section 3.1. One starts with supergravity coupled to the desired number of chiral and vector multiplets, having the desired interactions. One then adds an extra set of self-interacting chiral multiplets with a convenient superpotential W, whose rôle will be to provide the background values ϕ_0, $\bar{\phi}_0$, to support, for example, $\mathbb{S}^4$ spaces. They couple to gravity, but they do not couple to the chiral and vector multiplets of the physical theory that one wishes to study. By construction, the Lagrangian is supersymmetric, since the background values ϕ_0, $\bar{\phi}_0$ solve the conditions for supersymmetry. Then one takes the limit where the Planck mass M_P goes to infinity, i.e. $\kappa \to 0$ by first rescaling $\psi_\mu \to \kappa \psi_\mu$ and with fixed $\kappa \phi_0$, $\kappa \bar{\phi}_0$.

More generally, to construct a supersymmetric theory on a curved d-dimensional space, one shall start with a suitable d-dimensional gauged supergravity with $N \leq 4$ supersymmetries, couple it to the desired matter multiplets, plus additional free matter multiplets whose rôle is to provide background that support the curved supersymmetric space. The detailed construction of Lagrangians in specific models is beyond the scope of this paper, which is about classification.

4 Supersymmetric spaces in five dimensions

We shall consider $N = 2$ gauged supergravity coupled to n_V vector multiplets and n_H hypermultiplets. One may also consider adding tensor multiplets, though for our purposes this is unnecessary. We recall that the fields of the $N = 2$ supergravity multiplet are the fünfbein e_μ^a, two gravitini ψ_μ^i ($i = 1, 2$) and a vector boson A_μ; the hypermultiplet contains four real scalars q and hyperinos ζ; the $N = 2$ vector multiplet has a vector field, two spin-1/2 fermions and one real scalar field. The fermions of each of these multiplets transform as doublets under the $SU(2)_R$ R-symmetry group of the $N = 2$ superalgebra. The ungauged theory is determined in terms of real symmetric tensor C_{IJK}, $I, J, K = 0, ..., n_V$. The vector multiplet scalars h^I satisfy $C_{IJK} h^I h^J h^K = 1$ which define an n_V dimensional hypersurface of scalars ϕ^x called a 'very special real' manifold.

We consider the supersymmetry variation of fermions after turning on constant background values for vector fields and scalars. Like in the four-dimensional case, the vector fields must be constant in

order to to satisfy the requirement $F_{\mu\nu}(A) = 0$ coming from integrability of the vanishing gravitino transformation. The supersymmetry transformations for the gravitino ψ_μ^i, gauginos λ^{xi} and hyperinos ζ^A take the following form (see e.g. [34, 35])

$$\delta\psi_\mu^i = \nabla_\mu\xi^i - g\kappa^2 A_\mu^I P_I^{ij}\xi_j - \frac{ig}{\kappa\sqrt{6}}\,\Gamma_\mu P^{ij}\xi_j \ ,$$

$$\delta\lambda^{xi} = -\frac{ig}{2}\Gamma^\mu A_\mu^I K_I^x\xi^i - \frac{g}{\kappa^2}P^{xij}\xi_j + \frac{g}{\kappa^2}W^x\xi^i \ , \qquad (4.1)$$

$$\delta\zeta^A = \frac{ig}{2}\Gamma^\mu A_\mu^I k_I^X f_X^{iA}\xi_i + \frac{g}{\kappa^2}\mathcal{N}_i^A\xi^i \ .$$

$i = 1, 2$, $\xi_j = \epsilon_{ij}\xi^j$. One can switch between $SU(2)$ and vector indices by using the relation

$$A_i{}^j \equiv i\vec{A} \cdot \vec{\sigma}{}^{\ j} \ , \qquad (4.2)$$

where $\vec{\sigma}$ are Pauli matrices. The spinors obey the pseudo Majorana condition $\bar{\xi}^i = (\xi_i)^*\Gamma_0 = \xi^{iT}C$. P^{ij} and W depend on the constant backgrounds for the scalars q^X and ϕ^x. For further notation and details we refer to [34, 35].

In the Euclidean theory, we treat ξ^i and $\bar{\xi}^i$ as independent spinors. We will define $\bar{\xi}^j = -i\xi_j$. The vanishing of the gravitino transformation then implies the two separate conditions

$$0 = \nabla_\mu\xi^i - ig\kappa^2 A_\mu^I P_I^{ij}\bar{\xi}^j + m\,\Gamma_\mu P^{ij}\bar{\xi}^j \ ,$$

$$0 = \nabla_\mu\bar{\xi}^i + ig\kappa^2 \bar{A}_\mu^I \bar{P}_I^{ij}\xi^j + m\,\Gamma_\mu \bar{P}^{ij}\xi^j \ , \qquad (4.3)$$

with

$$m \equiv \frac{g}{\kappa\sqrt{6}} \ . \qquad (4.4)$$

It should be noted that in particular cases the present Killing spinor equation simplifies. For example, in gauged supergravity coupled to only vector multiplets described in [34] one has $P^{ij} \propto \delta^{ij}$.

Let us first consider solutions with $A_\mu^I = 0$. The gaugino and hyperino equations give constraints on the values of the constant scalar fields. Supersymmetric solutions with no restrictions on $\xi^i, \bar{\xi}^i$ require that

$$P^{xij} = W^x = \mathcal{N}_i^A = 0 \ . \qquad (4.5)$$

These conditions are very similar to the ones appearing in studies of supersymmetric renormalization flows in AdS solutions of $N = 2$ 5d gauged supergravity [36]. The solution, which in particular depends on the specific scalar manifolds, represents fixed points of the renormalization group where the scalars are frozen. For the purpose of this work, one just needs to bear in mind that the possible constant values

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of the scalar fields will be given by the solutions to (4.5), which is to be found explicitly once the model is specified.

The scalar potential then simplifies to

$$V = -\frac{4g^2}{\kappa^4} \vec{P} \cdot \vec{\bar{P}} \tag{4.6}$$

where we assumed $[P, \bar{P}] = 0$. Let us now consider the gravitino equations. The integrability conditions give

$$0 = \big[\nabla_\nu, \nabla_\mu\big]\xi^i + \kappa^2 \frac{V}{12}\Gamma_{\mu\nu}\, \xi^i \ ,$$
$$0 = \big[\nabla_\nu, \nabla_\mu\big]\bar{\xi}^i + \kappa^2 \frac{V}{12}\Gamma_{\mu\nu}\, \bar{\xi}^i \ . \tag{4.7}$$

By a similar calculation as in the previous section, we get the conditions

$$W_{\mu\nu\rho\sigma} = 0 \ , \qquad R_{\mu\nu} = \kappa^2 \frac{2V}{3}g_{\mu\nu} \ , \tag{4.8}$$

i.e. the space is Einstein-Weyl. This implies that the space is locally isometric to $\mathbb{H}_5$ or $\mathbb{S}^5$ according to the case, $V < 0$ or $V > 0$, where V is to be evaluated at the scalar background. The space $\mathbb{S}^5$ has been used to carry out exact calculations of the partition function in $N = 1$ Super Yang-Mills theory and superconformal theories [8, 12].

Spaces of the form $X_4 \times \mathbb{S}^1$ arise by turning on constant Wilson lines. Define $t^{ij} = \kappa^2 A_5^I P_I^{ij}$, $\bar{t}^{ij} = \kappa^2 \bar{A}_5^I \bar{P}_I^{ij}$. Similarly to the four-dimensional case, we demand

$$\partial_5 \xi^i = \partial_5 \bar{\xi}^i = 0 \ , \tag{4.9}$$

so that

$$igt^{ij}\bar{\xi}^j = m\Gamma_5 P^{ij}\bar{\xi}^j \ , \qquad ig\bar{t}^{ij}\xi^j = -m\Gamma_5 \bar{P}^{ij}\xi^i \ . \tag{4.10}$$

The presence of Γ_5 reduces the number of supersymmetries by a factor $1/2$. To have no further restrictions on the spinors, we impose the following algebraic equations for the background values of scalars and Wilson lines:

$$igt^{ij} = mP^{ij} \ , \qquad ig\bar{t}^{ij} = -m\bar{P}^{ij} \ . \tag{4.11}$$

The remaining equations are

$$\nabla_{\hat{\mu}}\xi^i = -m\Gamma_{\hat{\mu}}P^{ij}\bar{\xi}^i \ ,$$
$$\nabla_{\hat{\mu}}\bar{\xi}^i = -m\Gamma_{\hat{\mu}}\bar{P}^{ij}\xi^i \ . \tag{4.12}$$

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The integrability condition then gives

$$W_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} = 0 , \qquad R_{\hat{\mu}\hat{\nu}} = 6m^2 P_k^j \bar{P}_j^k g_{\hat{\mu}\hat{\nu}} = \frac{\kappa^2}{2} \, V \, g_{\hat{\mu}\hat{\nu}} , \qquad (4.13)$$

i.e. we get a space locally isometric to $\mathbb{S}^4 \times \mathbb{S}^1$ or $\mathbb{H}^4 \times \mathbb{S}^1$ according to the sign of V.

It is interesting to compare with [11], where supersymmetric gauge theories on $\mathbb{S}^4 \times \mathbb{S}^1$ are discussed. In this work, maintaining supersymmetry required introducing by hand a new contribution in the Killing spinor equation containing a symmetric tensor t^{ij}. However, the construction of a supersymmetric Yang-Mills action turns out to be problematic. Despite some similarities, in the present approach the Killing spinor equation has a structure which is different from the one proposed in [11]. Our approach explains the origin of the tensor t^{ij} and justifies this term in the gravitino transformation laws. Moreover, since the supergravity action coupled to vector multiplets is, by construction, supersymmetric, we expect that our approach also prescribes how to determine the complete supersymmetric and gauge invariant Yang-Mills action. We leave this interesting problem for future research.

5   Supersymmetric spaces in six dimensions

According to the group theoretic analysis of section 2, six dimensions is the highest dimension that can be considered for a consistent quantum theory with global supersymmetry based on simple supergroups. Consequently, it is the highest dimension allowed from AdS/CFT correspondence, an example being the (2,0) superconformal field theory describing the low energy dynamics of M5 branes.

5.1 $F(4)$ gauged Supergravity

In order to find possible supersymmetric spaces in six dimensions, we will employ Romans $F(4)$ gauged supergravity [25]. The theory includes an $SU(2)$ connection A_μ^{ij}, an abelian connection a_μ and an antisymmetric form $B_{\mu\nu}$, with field strengths $F_{\mu\nu}(A)$, $f_{\mu\nu}(a)$ and $G_{\mu\nu\rho}$, respectively. We consider configurations with $F_{\mu\nu}(A) = G_{\mu\nu\rho} = f_{\mu\nu} = 0$ and we will allow only for a non-vanishing flat $SU(2)$ field A_μ^{ij}. In Lorentzian signature, fermions in the theory are symplectic Majorana. However, in Euclidean signature the symplectic Majorana condition is relaxed. The gravitino transformation law may be written as

$$\delta\psi_\mu^i = \nabla_\mu \xi^i - ig A_{\mu\ j}^{\ i} \xi^j + T \, \Gamma_\mu \Gamma_7 \xi^i , \qquad (5.1)$$

where

$$T = -\frac{1}{8\sqrt{2}}(g \, e^{\phi/\sqrt{2}} + m \, e^{-3\phi/\sqrt{2}}) . \qquad (5.2)$$

Note that the scalar ϕ should be constant as follows from the vanishing of supersymmetric shifts of the four spin-$\frac{1}{2}$ fields χ_i in the gravity multiplet

$$\delta\chi_i = \frac{1}{\sqrt{2}}\Gamma^\mu\partial_\mu\phi\xi_i + \frac{1}{4\sqrt{2}}\left(g\,e^{\phi/\sqrt{2}} - 3m\,e^{-3\phi/\sqrt{2}}\right)\Gamma_7\,\xi_i \ . \tag{5.3}$$

Then $\delta\chi_i = 0$ gives that

$$\phi = \frac{\sqrt{2}}{4}\ln\left(\frac{3m}{g}\right)\ , \quad T = -\frac{1}{6\sqrt{2}}g\left(\frac{3m}{g}\right)^{1/4}\ . \tag{5.4}$$

There are now two cases:

I. $A_\mu{}^i{}_j = 0$

The integrability condition of (5.1) is written here as

$$0 = \left[\nabla_\nu, \nabla_\mu\right]\xi^i - T\bar{T}\Gamma_{\mu\nu}\,\xi^i\ , \tag{5.5}$$

and leads to

$$W_{\mu\nu\rho\sigma} = 0\ , \quad R_{\mu\nu} = -3T\bar{T}\,g_{\mu\nu}\ . \tag{5.6}$$

The only solution to (5.6) is the round $\mathbb{S}^6$ (in the compact case, $T\bar{T} < 0$) or $\mathbb{H}^6$ (in the non-compact case, $T\bar{T} > 0$). For $\mathbb{S}^6$, the background enjoys an $SO(7)$ isometry and, for $\mathbb{H}^6$, $SO(6,1)$. The $SO(6,1)$ can be the bosonic part of a supersymmetry algebra whereas, according to Nahm's classification, $SO(7)$ cannot. Therefore $\mathbb{S}^6$ cannot support supersymmetry.

In Lorentzian space, the $F(4)$ gauged supergravity theory has anti-de Sitter solutions with $SO(5,2)\times SU(2)$ bosonic symmetry, representing a subgroup of $F(4)$. This is of course in Nahm's list. Nahm's classification also includes $SO(7)\times SO(2,1)$ – case **XII** in section 2 – which is another real form for the bosonic subgroup of $F(4)$. [3] Interpreting this $SO(7)$ as the symmetry of $\mathbb{S}^6$ implies that $SO(2,1) = SU(1,1)$ must arise as an internal R-symmetry. This group is non-compact and would inevitably lead to ghosts. We shall expand on the problems of S^6 in section 5.2 and section 6.

II. $A_\mu{}^i{}_j \neq 0$

In this case, we may consider Wilson lines for an Abelian subgroup of the R-symmetry group $SU(2)$. For example, in the simplest case, we may switch on a a flat $A = A_6{}^i{}_j dx^6 = \frac{1}{2}A_6(\sigma_3)^i_j dx^6\ U(1)$ field. Then, the vanishing of the gravitino shifts can be written as

$$0 = \nabla_6\xi^i - igA_6(\sigma_3)^i_j\xi^j + T\ \Gamma_6\Gamma_7\xi^i\ ,$$
$$0 = \nabla_{\hat{\mu}}\xi^i + T\ \Gamma_{\hat{\mu}}\Gamma_7\xi^i\ , \quad \hat{\mu} = 1, ..., 5\ . \tag{5.7}$$

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[3]The other real forms of $F(4)$ are $SU(1,1)\times SO(7)$, $SU(2)\times SO(6,1)$ and $SU(2)\times SO(4,3)$ [28]. We thank Paul Sorba for clarification on this point.

These equations are solved then by $\xi^i = \xi^i(x^{\hat{\mu}})$ on $\mathbb{S}^1 \times \mathbb{S}^5$. The corresponding superalgebra is $su(4|2)$, described by case **II** of section 2.

5.2 Supersymmetric algebras in 6d

One would like then to know why $\mathbb{S}^6$ fails to admit supersymmetry. To understand this, let us consider Killing spinors η_i which satisfy equations of the general form

$$\nabla_\mu \eta_i + T\,\Gamma_\mu \Gamma_7 \eta_i = 0\,, \quad i = 1, 2\,. \tag{5.8}$$

Let us denote by η_i^I ($I, J, ... = 1, ..., n$) the n-independent solutions to the above equation (5.8). Using the Killing spinors, we may define the vectors

$$V_\mu{}^{IJ} = \bar{\eta}_i^I \Gamma_\mu\, \eta_j^J\, \epsilon^{ij}\,, \quad U_\mu{}^{IJ} = \bar{\eta}_i^I \Gamma_\mu\, \eta_j^J\, \delta^{ij}\,. \tag{5.9}$$

In the particular case of $\mathbb{S}^6$ we are interested in, we have $I, J = 1, 2, 3, 4$ (corresponding to the $\mathbf{4}, \bar{\mathbf{4}}$ representations of $SO(6)$). It is obvious that $V_\mu{}^{IJ}$ and $U_\mu{}^{IJ}$ are symmetric and antisymmetric, respectively

$$V_\mu{}^{IJ} = V_\mu{}^{JI}\,, \quad U_\mu{}^{IJ} = -U_\mu{}^{JI}\,, \tag{5.10}$$

in their $SO(6)$ spinorial indices. In other words, $V_\mu{}^{IJ}$ belongs to the $\mathbf{15}$ and $U_\mu{}^{IJ}$ to the $\mathbf{6}$ dimensional representation of $SO(6)$. It is also easy to see, by using (5.8), that $V_\mu{}^{IJ}, U_\mu{}^{IJ}$ are Killing vectors for pure imaginary T. Indeed,

$$\nabla_\mu V_\nu{}^{IJ} = 2\mathrm{Re}\,T\, g_{\mu\nu}\, \bar{\eta}_i^I\, \Gamma_7\, \eta_j^J\, \epsilon^{ij} + 2i\mathrm{Im}T\, \bar{\eta}_i^I\, \Gamma_{\mu\nu}\Gamma_7\, \eta_j^J\, \epsilon^{ij}\,,$$
$$\nabla_\mu U_\nu{}^{IJ} = 2\mathrm{Re}T\, g_{\mu\nu}\, \bar{\eta}_i^I\, \Gamma_7\, \eta_j^J\, \delta^{ij} + 2i\mathrm{Im}T\, \bar{\eta}_i^I\, \Gamma_{\mu\nu}\Gamma_7\, \eta_j^J\, \delta^{ij}\,, \tag{5.11}$$

and therefore for $\mathrm{Re}T = 0$, we have

$$\nabla_\mu V_\nu{}^{IJ} + \nabla_\nu V_\mu{}^{IJ} = 0\,, \quad \nabla_\mu U_\nu{}^{IJ} + \nabla_\nu U_\mu{}^{IJ} = 0\,. \tag{5.12}$$

Thus, the twenty one vectors $k_\mu^{AB} = (V_\nu{}^{IJ}, U_\nu{}^{IJ})$ ($A, B = 1, ... 7$ and $k_\mu^{AB} = -k_\mu^{BA}$) generate the $SO(7)$ isometry of the $\mathbb{S}^6$ and transform under the $\mathbf{21} = \mathbf{6} + \mathbf{15}$ dimensional representation of $SO(7)$.

There is also a global symmetry since the Killing spinors η_i^I carry a global index i as well. It is easy to see that the vectors $U_\mu{}^{IJ}$ and $V_\mu{}^{IJ}$ are invariant under $USp(2) = SU(2)$ rotations of the spinors η_i^I. The full symmetry is $SO(7) \times SU(1, 1)$, where the $SU(1, 1)$ generators can be taken to be $T_\pm = J_1 \pm iJ_2$, $T_3 = J_3$, with J_1, J_2, J_3 being the $SU(2)$ generators. This symmetry would be the bosonic (even)

$\mathcal{G}_0$ part of supersymmetry. Then, there should also exist an odd part $\mathcal{G}_1$, transforming in the spinorial representation of the even (bosonic) part of the supersymmetry algebra. To find $\mathcal{G}_1$, we recall that

$$\{\mathcal{G}_1, \mathcal{G}_1\} \subset \mathcal{G}_0 . \tag{5.13}$$

As we are looking for supersymmetry, the odd generators should be fermionic and so in the spinorial representation of the even $SO(7) \times SU(1,1)$ algebra. To make things simpler, we will consider first the case where the even part is just $SO(7)$. In this case, and since

$$\mathbf{8} \times \mathbf{8} = \mathbf{1_s} + \mathbf{21_a} + \mathbf{7_a} + \mathbf{35_s} , \tag{5.14}$$

the fermionic anticommutator should close into the $\mathbf{21}$ (i.e., the generators M_{mn} of the $SO(7)$),

$$\{Q_\alpha, Q_\beta\} = \kappa(\sigma^{mn}\mathcal{C}^{-1})_{\alpha\beta}M_{mn} , \tag{5.15}$$

where $(m,n,... = 1,...,7)$, $(\alpha,\beta,... = 1,...8)$, κ is an appropriate constant and $\sigma^{mn} = \frac{1}{4}[\gamma^m, \gamma^n]$ is the $SO(7)$ spinorial representation[4]. The generators M_{mn} satisfy the $SO(7)$ algebra

$$[M_{mn}, M_{kl}] = -\delta_{mk}M_{nl} + \delta_{ml}M_{nk} - \delta_{nl}M_{mk} + \delta_{nk}M_{ml} . \tag{5.17}$$

Since Q_α are fermions, they transform under the spinorial representation as

$$[Q_\alpha, M_{mn}] = \frac{1}{2}(\sigma_{mn})_\alpha{}^\beta Q_\beta . \tag{5.18}$$

Now, all commutation relations have been defined and what remains to be checked is the Jacobi identity. For the triplet $(M_{mn}, Q_\alpha, Q_\beta)$ it reads

$$[\{Q_\alpha, Q_\beta\}, M_{mn}] + \{[M_{mn}, Q_\alpha], Q_\beta\} + \{[M_{mn}, Q_\beta], Q_\alpha\} = 0 , \tag{5.19}$$

and leads to the conditions

$$\begin{aligned} 0 = {} & \kappa(\sigma_{kl}\mathcal{C}^{-1})_{\alpha\beta}\Big(\delta_{mk}M_{nl} - \delta_{ml}M_{nk} + \delta_{nl}M_{mk} - \delta_{nk}M_{ml}\Big) \\ & - \frac{1}{2}(\sigma_{mn})_{\beta\gamma}(\sigma^{kl}\mathcal{C}^{-1})_{\gamma\alpha}M_{kl} - \frac{1}{2}(\sigma_{mn})_{\alpha\gamma}(\sigma^{kl}\mathcal{C}^{-1})_{\gamma\beta}M_{kl} . \end{aligned} \tag{5.20}$$

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[4] Standard Poincaré supersymmetry corresponds to the closure of the fermionic anticommutator in the $\mathbf{7}$ representation of $SO(7)$, i.e.,

$$\{Q_\alpha, Q_\beta\} = (\gamma^m\mathcal{C}^{-1})_{\alpha\beta}P_m . \tag{5.16}$$

Obviously, this relation is the same for any group $SO(d)$ with odd part in the spinorial representation. It is also clear that such a relation cannot be satisfied in general and it can only be valid accidentally. This is indeed the case for the $SO(7)$ group. By using the following representation of the $SO(7)$ 8×8 γ-matrices

$$(\gamma_m)_{ab} = i\psi_{mab} , \quad (\gamma_m)_{8a} = i\delta_{ma} \tag{5.21}$$

where ψ_{mab} are the octonionic structure constants [37] and the relation

$$\psi^{abc}\psi_{dhc} = \delta_d^a \delta_h^b - \delta_d^b \delta_h^a - \frac{1}{3!}\epsilon^{ab}{}_{dgijk}\psi^{ijk} , \tag{5.22}$$

we find that miraculously (5.20) is satisfied. However, there are also other Jacobi identities which should be satisfied. Among these, it is straightforward to check that

$$[\{Q_\alpha, Q_\beta\}, Q_\gamma] + [\{Q_\alpha, Q_\gamma\}, Q_\beta] + [\{Q_\beta, Q_\gamma\}, Q_\alpha] = 0 , \tag{5.23}$$

fails to be satisfied. The Jacobi identity can be satisfied if $SO(7)$ is extended in an appropriate way. In particular, the appropriate extension turns out to be $SO(7) \times SU(1,1)$. In this case it can be proven that a superalgebra exists [26, 38, 39]. It is defined by the commutation relations

$$[T_i, T_j] = ic_{ij}^k T_k , \qquad\qquad [T_i, M_{mn}] = 0 , \tag{5.24}$$

$$[M_{mn}, M_{kl}] = -\delta_{mk}M_{nl} + \delta_{ml}M_{nk} - \delta_{nl}M_{mk} + \delta_{nk}M_{ml} , \tag{5.25}$$

$$[T_i, Q_\alpha^a] = \frac{1}{2}(\tau_i)^a{}_b Q_\alpha^b , \qquad [M_{mn}, Q_\alpha^a] = (\sigma_{mn})_\alpha{}^\beta Q_\beta^a , \tag{5.26}$$

$$\{Q_\alpha^a, Q_\beta^b\} = 2C_{\alpha\beta}^{(8)}(C^{(2)}\tau^i)^{ab}T_i + \frac{2}{3}C^{(2)ab}(C^{(8)}\sigma^{mn})M_{mn} , \tag{5.27}$$

where $(i = 1,2,3)$, (τ^i) are the fundamental representation of $SU(1,1) = SO(2,1)$ and $C^{(2)}(= i\tau^2)$, $C^{(8)}$ are the 2×2 and 8×8 charge conjugation matrices. Of course, we recognize here the exceptional $F(4)$ superalgebra. Note that the odd generators of the algebra are in the $(\mathbf{8}, \mathbf{2})$ representation of the even $SO(7) \times SU(1,1)$ and therefore it is a supersymmetry algebra. This is case **XII** in section 2. There is another extension by which the Jacobi identity can be satisfied, namely the $osp(7|2)$ superalgebra, also with bosonic group $SO(7) \times SU(1,1)$, corresponding to case **X**. However, in this case the odd generators are not in the spinorial representation of the $SO(7)$ isometry group. For the $F(4)$ superalgebra, the odd generators are in the spinorial representation of $SO(7)$ but one still has the problem that the R-symmetry group is the non-compact $SU(1,1)$, which has indefinite metric (Cartan-Killing metric of signature 1) and therefore any theory invariant under $F(4)$ supersymmetry will necessarily have ghosts. This is the

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reason why the $F(4)$ of case **XII** cannot be used as a possible superalgebra in a ghost-free supersymmetric theory on $\mathbb{S}^6$.

One may also ask if there are *superconformal* theories on $\mathbb{S}^6$. If that was the case, there should be a superalgebra with even part containing $SO(7,1)$. However, a simple inspection of the classification of possible superalgebras reveals that this is not possible. Therefore, $\mathbb{S}^6$ does not admit superconformal theories and of course, this is also the case for Euclidean $\mathbb{R}^6$, which should share the same $SO(7,1)$ symmetry.

It is instructive to compare with the lower dimensional cases where we have supersymmetry. For example, let us now explicitly demonstrate why $\mathbb{S}^5$ does admit supersymmetry. The isometry group of $\mathbb{S}^5$ is $SO(6)$ and we can consider the generators T_μ^ν (in the **15** of $SO(6) \simeq SU(4)$) to satisfy

$$[T_m^n, T_k^l] = \delta_k^n T_m^l - \delta_m^l T_k^n \; , \tag{5.28}$$

where $m, n, k, l = 1, 2, 3, 4$. We can take the odd part of the superalgebra of which $SO(6)$ is the even part to be generated by Q_m and Q^n in the **4** and $\bar{\mathbf{4}}$ spinorial representations of $SO(6)$,

$$[T_m^n, Q_k] = \delta_k^n Q_m \; , \quad [T_m^n, Q^k] = -\delta_m^k Q^n \; . \tag{5.29}$$

Then, since

$$\mathbf{4} \times \mathbf{4} = \mathbf{6} + \mathbf{10} \; , \quad \mathbf{4} \times \bar{\mathbf{4}} = \mathbf{1} + \mathbf{15} \; , \tag{5.30}$$

we see that necessarily

$$\{Q^m, Q^n\} = 0 \; , \quad \{Q_m, Q_n\} = 0 \; , \quad \{Q^m, Q_n\} = \beta T_n^m + \delta_n^m Z \; , \tag{5.31}$$

where Z is an $SO(6)$ singlet and β a constant, which is specified from Jacobi identity to be $\beta = 0$. Then the operators T_n^m, Z, Q_m, Q^n generate the supergroup $su(4|1)$ and, since the odd generators are in the spinorial representation of the even $SO(6) \times U(1)$ algebra, it is a supersymmetric algebra. Therefore, we see that the existence of supersymmetry on $\mathbb{S}^5$ is basically due to the Lie algebra isomorphism $SO(6) \simeq SU(4)$, which permits that $SO(6)$ also arises in the $su(n|N)$ series of supergroups.

6   Supersymmetry in $d > 5$

According to the discussion in section 2 and above, it is not possible to have supersymmetry on d-spheres with $d > 5$. Field theories with rigid supersymmetry exist up to $d = 10$. One important case that can be

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ruled out immediately is supersymmetry on S^{10}, due to the fact that the minimal spinor representation for $SO(11)$ is 32. In particular, $N = 1$ 10d super Yang-Mills theory cannot be put on S^{10}.

More generally, in $d > 5$, one of the problems is that, as explained, the Jacobi conditions (5.20), (5.23) fail to be satisfied for odd generators in the spinorial representation of the even ($SO(d)$ (or $SO(d+1,1)$) part, unless the bosonic symmetry is extended in an appropriate way. To identify the odd part of the superalgebra, let us assume that the corresponding generators Q_α transform in a particular representation Δ as

$$\{Q_\alpha, Q_\beta\} = \kappa\left(\Delta^{\mu\nu}\right)_{\alpha\beta} M_{\mu\nu} \, , \tag{6.1}$$

$$[Q_\alpha, M_{\mu\nu}] = \frac{1}{2}\left(\Delta_{\mu\nu}\right)_\alpha{}^\beta Q_\beta \, . \tag{6.2}$$

Then, the Jacobi identity (5.20) gives

$$\begin{aligned}
0 =& \ \kappa\left(\Delta_{\kappa\lambda}\right)_{\alpha\beta}\left(\delta_{\mu\kappa} M_{\nu\lambda} - \delta_{\mu\lambda} M_{\nu\kappa} + \delta_{\nu\lambda} M_{\mu\kappa} - \delta_{\nu\kappa} M_{\mu\lambda}\right) \\
&- 2\left(\Delta_{\mu\nu}\right)_{\beta\gamma}\left(\Delta^{\kappa\lambda}\right)_{\gamma\alpha} M_{\kappa\lambda} - \left(\Delta_{\mu\nu}\right)_{\alpha\gamma}\left(\Delta^{\kappa\lambda}\right)_{\gamma\beta} M_{\kappa\lambda} \, .
\end{aligned} \tag{6.3}$$

The solution to this condition specifies the allowed representation for the odd generators. It is easy to check that (6.3) is solved for

$$\left(\Delta_{\mu\nu}\right)_{\alpha\beta} = \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha} \, , \tag{6.4}$$

and $\kappa = 1$. Thus, $(M_{\mu\nu}, Q_\alpha)$ form a superalgebra if Q_α transforms in the vectorial representation of the even $SO(d)$ part. This superalgebra is the $osp(d|2)$. This appears as case **X** in section 2, except for the crucial difference that here $SO(d)$ appears as the isometry and $SO(2,1)$ as the R-symmetry, whereas in section 2 is the opposite. Because for $d > 5$ the odd generators are not in the spinorial representation of the isometry group, $osp(d|2)$ is not a supersymmetry algebra (and, similarly, $osp(d+1,1|2)$ is not a superconformal algebra). Moreover, the R-symmetry is represented by the non-compact $SO(2,1)$ group, therefore invariant Lagrangians will contain ghosts. We emphasize the double role that the even part $SO(d) \times SO(2,1)$ of $osp(d|2)$ can play: either $SO(2,1)$ is the isometry and $SO(d)$ the R-symmetry, or the opposite, $SO(d)$ is the isometry and $SO(2,1)$ the R-symmetry. Clearly, only the first case makes sense (**X** of section 2) if one wishes to have a ghost-free theory.

In general, for any m, n, the even part of the $osp(m|n)$ is $SO(m) \times Sp(n)$ and the odd part is in the (m, n) representation of $SO(m) \times Sp(n)$. Therefore it might seem that these difficulties would also apply to lower dimensions. However, this is not true. In fact, (5.20) holds for certain specific cases. In particular, (5.20) 'accidentally' holds in $d \leq 5$ and the reason lies on the various Lie algebra isomorphisms of the

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orthogonal groups with unitary and symplectic groups. Although orthogonal $SO(d)$ algebras are expected in the $osp(d|2)$ superalgebras which are not supersymmetries, some of them, due to isomorphisms, also appear in $su(d|N)$ superalgebras. These isomorphisms are

$$SO(2) \simeq U(1)\,, \quad SO(3) \simeq SU(2)\,, \quad SO(4) \simeq SU(2) \times SU(2)\,,$$

$$SO(5) \simeq USp(4)\,, \quad SO(6) \simeq SU(4)\ ,$$

$$SO(5,1) \simeq SU^*(4)\,, \quad SO(4,1) \simeq USp(2,2)\,, \quad SO(3,1) \simeq SL(2,\mathbb{C})\,, \quad SO(2,1) \simeq SU(1,1)\ , \quad (6.5)$$

and allow for the existence of supersymmetries on d-spheres with $d \leq 5$. The missing case $SO(6,1)$ is not due to some isomorphism but rather, as noticed already, due to the fact that $SO(6,1) \times SU(2)$ is a real form of $F(4)$.

In conclusion, Euclidean field theories with rigid supersymmetry cannot be consistently defined on round spheres $\mathbb{S}^d$ if $d > 5$. In particular, in $d = 6$ a superalgebra exists but the R-symmetry is non-compact leading to ghosts states. Superconformal theories cannot be consistently defined on Euclidean $\mathbb{S}^d$ with $d > 5$, nor on any space conformal to $\mathbb{S}^d$ such as Euclidean $\mathbb{R}^d$ or $\mathbb{S}^{d-1} \times \mathbb{S}^1$ with $d > 5$.

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